THE SECOND CENTRALIZER OF A BERNOULLI SHIFT IS JUST ITS POWERS

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ABSTRACT

If $\mathcal{F}$ is a collection of measure preserving transformations of a probability space, by $C(\mathcal{F})$, the centralizer of $\mathcal{F}$, we mean the group of all measure preserving transformations S such that $TS = ST$ for all $T \in \mathcal{F}$. We show here that if T is a Bernoulli shift, then $C(C(T)) = \{T^i \mid i \in \mathbf{Z}\}$. The proof is carried out by constructing an action of $\mathbf{Z}^2$, $\{T_1^i \circ T_2^j \mid i, j \in \mathbf{Z}\}$, where T_1 is a Bernoulli shift of arbitrary entropy, but for any $j \neq 0$, $C(\{T_1, T_2^j\}) = \{T_1^l \circ T_2^k \ l, k \in \mathbf{Z}\}$. The construction is a two-dimensional analogue of Ornstein's "rank one mixing" transformation.

I. Introduction

If $\mathcal{T}$ is a collection of measure preserving transformations of a probability space, by $C(\mathcal{T})$, the centralizer of $\mathcal{T}$, we mean the group of all measure preserving transformations S such that $TS = ST$ for all $T \in \mathcal{T}$. What we want to show is that if T is a Bernoulli shift then

$$C(C(T)) = \{T^i \mid i \in Z\}. \tag{1.1}$$

The way we will show this is by constructing explicitly a Z^2 action generated by two maps, T_1 and T_2, where T_1 is a Bernoulli shift of arbitrary entropy and for any $j \neq 0$,

$$C(\{T_1, T_2^j\}) = \{T_1^l \circ T_2^k \mid l, k \in Z\}. \tag{1.2}$$

As $\{T_1^l \circ T_2^k \mid l, k \in Z\} \subset C(T_1)$, this implies $C(C(T_1)) \subset \{T_1^l \circ T_2^k \mid l, k \in Z\}$. This reduces $C(C(T_1))$ almost to what we want. All that remains is to show that for no $j \neq 0$ is $T_2^j \in C(C(T_1))$. As T_1 is a Bernoulli shift it has a square root $T_1^{1/2} \in C(T_1)$. If $T_2^j \in C(C(T_1))$ then $T_1^{1/2} \in C(\{T_1, T_2^j\})$ conflicting with (1.2).

Ornstein in [1] and Polit in [4] have shown how to build a Z action T so that $C(\{T^j\}) = \{T^i \mid i \in Z\}$, for any $j \neq 0$. We will borrow their method and extend it to a two-dimensional construction. If (1.2) were all we asked, the construction

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would be quite easy. We ask also, though, that T_1 be a Bernoulli process whose entropy is at our disposal. This last condition especially adds complexity to the construction.

II. Building the process

Our construction will take the form of an increasing sequence of rectangular arrays A_n of symbols from $P = \{0, S, B\}$. The $(n+1)$-th array will be built out of many copies of the n-th array, situated in various positions in a matrix of spacers S. These arrays convert in a standard manner to a two-dimensional process (T_1, T_2, P) on which the T_1, T_2, P-name of a point is a nested sequence of copies of these arrays.

We will now describe inductively how to build the A_n. The array A_0 is an $h_1(0)$ by $h_2(0)$ rectangle of 0's. The parameters $h_1(0)$ and $h_2(0)$ will be fixed later to control the entropy of T_1. Now suppose we have built A_n, with dimensions $h_1(n)$ by $h_2(n)$. To build A_{n+1} we will need some further parameters. The first of these are $S_1(n+1)$, $S_2(n+1)$ and $N_1(n+1)$. We will set their values later. For any $0 \leqq \sigma_1 \leqq S_1(n+1)$ and $0 \leqq \sigma_2 \leqq S_2(n+1)$, $A_n^{\sigma_1, \sigma_2}$ is a copy of A_n with a border of σ_1 S's on its left, $(S_1(n+1) - \sigma_1)$ S's on its right, σ_2 S's on top and $(S_2(n+1) - \sigma_2)$ S's on bottom. This we will call an $(n+1)$-box with an A_n in it. Let $L(n+1)$ be the set of all sequences $l = \{(\sigma_1^1, \sigma_2^1), (\sigma_1^2, \sigma_2^2), \cdots, (\sigma_1^{N_1(n+1)}, \sigma_2^{N_1(n+1)})\}$ of length $N_1(n+1)$ from $S_1(n+1) \times S_2(n+1)$. Now

$$A_n^l = A_n^{\sigma_1^1, \sigma_2^1} \circ A_n^{\sigma_1^2, \sigma_2^2} \circ \cdots \circ A_n^{\sigma_1^{N_1(n+1)}, \sigma_2^{N_1(n+1)}},$$

i.e., a row of $N_1(n+1)$ $(n+1)$-boxes, each containing an A_n, the j-th situated at position σ_1^j, σ_2^j. Notice that fixing a name across k boxes, the $(k+1)$-th box can be any $A_n^{\sigma_1, \sigma_2}$, allowing a certain independence of the T_1 past. We will see later that if $S_1(n+1)$ and $S_2(n+1)$ are large enough, this will make T_1 a Bernoulli shift.

Now let $\mathscr{L}(n+1) = \{L_1(n+1) \cdots L_s(n+1)\}$, $s = (S_1(n+1)S_2(n+1))^{N_1(n+1)}!$, be all possible orderings of the elements of $L(n+1)$, and now

$$A_{n+1}^j = \begin{matrix} A_n^{l_1} \\ \circ \\ A_n^{l_2} \\ \circ \\ \vdots \\ \circ \\ A_n^{l_w} \end{matrix}$$

where $L_j = (l_1, l_2, \cdots, l_u)$, $u = (S_1(n+1)\,S_2(n+1))^{N_1(n+1)}$. Now, finally, to build A_{n+1}, we choose a parameter $N_2(n+1)$ and sequence $L_{j_1}, L_{j_2}, \cdots, L_{j_{N_2(n+1)}}$ of elements from $\mathscr{L}(n+1)$. A_{n+1} is

$$\begin{array}{c} A^{j_1}_{n+1} \\ \circ \\ A^{j_2}_{n+1} \\ \circ \\ \vdots \\ \circ \\ A^{j_{N_2(n+1)}}_{n+1} \end{array}$$

with a row of b's on the left side and bottom.

We use this method of concatenating A_n's vertically because it guarantees we see in any span of $2^{16(n+1)}(S_1(n+1)S_2(n+1))^{N_2(n+1)}$ rows of $(n+1)$-boxes in A_{n+1}, all possible A^l_n's distributed with $1/2^{16(n+1)}$ of uniformly. This will allow $S_2(n)$ to grow fast enough to make T_1 Bernoulli without growing so fast as to force infinite entropy. If we simply concatenated a "random" sequence of A^l_n's, the central limit theorem says we would need to scan the square of this many rows to get such a good distribution. This is too large a value for $S_2(n)$ to give both properties.

At this point the only conditions we put on $S_1(n)$, $S_2(n)$, $N_1(n)$ and $N_2(n)$ are that

$$\frac{S_1(n)}{h_1(n-1)} < \frac{1}{2^{16n}} \quad \text{and} \quad \frac{S_2(n)}{h_2(n-1)} < \frac{1}{2^{16n}} \tag{2.1}$$

to ensure T_1 and T_2 are defined on a probability space.

III. Choosing the parameters so that T_1 is Bernoulli of arbitrary entropy

As we indicated earlier, it is necessary to exercise a little care at this point as the properties we want for T_1 can be antagonistic. We have a further small difficulty in that T_1 has no obvious generating partition. To verify T_1 is Bernoulli we will take a sequence of partitions P_l which refine to all of the σ-algebra of measurable sets and verify (T_1, P_l) is very weak Bernoulli for all l and then apply Ornstein's Monotone Theorem. The partition P_l will be the division of Ω into sets of ω whose (T_1, T_2, P) name has zero point at the same coordinates in A_l, and the complement of these sets. Certainly $\bigvee_{i=0}^{\infty} P_l$ is the σ-algebra of measurable sets and $P_{l+1} \supset P_l$.

Now we require that

$$S_1(n+1) \geqq 2^{16n}(h_1(n-1)+S_1(n)) \quad \text{and} \tag{3.1}$$

$$S_2(n+1) \geqq 2^{16n}(S_1(n)S_2(n)^{N_1(n)})(h_2(n-1)+S_2(n)), \tag{3.2}$$

for n large enough. These are explicitly how fast $S_1(n)$ and $S_2(n)$ need to grow to guarantee T_1 is Bernoulli. To keep (2.1) we, thus, need $N_1(n)$, $N_2(n) > 2^{32n}$. To verify that (T_1, P_l) is v.w.b., we must consider what a future T_1, P_l distribution looks like on a fixed past atom. We begin this analysis as follows. Let B be a fixed T_1, P_l name across the first k n-boxes in A^l_n, $n \geqq l+2$. Let $\bar{B}$ be a possible continuation of B across the $(k+1)$-th n-box in A_n.

Lemma 3.1. *The distribution of possible $\bar{B}$, for all but 2^{-8n} of the B is any name from across A_{n-1} situated at any position $0 \leqq \sigma_1 \leqq S_1(n)$, each such occurring with within a fraction 2^{-16n} of the same density.*

Proof. Ignore those B which occur more than 2^{-8n} of the time within $S_2(n)$ of the top and bottom of the n-boxes. This is at most 2^{-8n} of the B. For any other, consider its occurrences at some fixed height in the boxes. Now the value $(\sigma_1^{k+1}, \sigma_2^{k+1})$ of the next box takes on any values up to $(S_1(n), S_2(n))$ each equally likely. By 3.2, the value $S_2(n)$ is large enough to scan at least $(2^{16n}-2)$ full cycles of types of rows A_{n-2}. Thus, each possible name across A_{n-1} is seen and with within a fraction 2^{-16n} of the same density.

Corollary 3.2. *For all but 2^{-8n} of the T_1, P_l names B across the first k boxes of A_n, $n \geqq l+2$, the distribution of possible names $\bar{B}$ across the remaining $N_1(n)-k$ boxes is within 2^{-16n} in $\bar{d}$ of the independent concatenation of the distribution of names across A_{n-1} situated in n-boxes independently at positions $0 \leqq \sigma_1 \leqq S_1(n)$.*

Proof. Follows from repeated application of Lemma 3.1 across the remaining $N_1(n)-k$ n-boxes.

Corollary 3.3. *Let B be a fixed past T_1, P_l name, ending after the first k boxes of A_n, $n \geqq l+2$. The distribution of possible future names $\bar{B}$ across the remaining $(N_1(n)-k)$ n-boxes, for all but 2^{-8n+1} of the B, is within 2^{-16n+1} in $\bar{d}$ of the independent concatenation of the distribution of T_1, P_l names across A_{n-1} situated independently at positions $0 \leqq \sigma_1 \leqq S_1(n)$ in n-boxes.*

Proof. Apply Lemma 3.1 inductively to ever longer past names across arrays, summing the errors.

Now we know what future names look like. It remains to show that $S_1(n)$ is large enough to get them all close in $\bar{d}$. Here we apply the "nesting argument" Ornstein used in [3] and Weiss applied to the equivalence theory in [5].

Let B_1 and B_2 be past atoms of (T_1, P_l). Suppose B_1 ends j_1 positions into the k_1-th n-box of A_n, and B_2 ends j_2 positions into the k_2-th n-box where $n \geqq l+2$ and $k_i \leqq N_1(n)(1-2^{-8n})$, and B_1 and B_2 extended across the remaining $h_1(n-1)+S_1(n)-j_1$ and $h_1(n-1)+S_1(n)-j_2$ coordinates in the k_1-th and k_2-th n-boxes are in the good set of Corollary 3.2. We want to match in $\bar{d}$ the future distributions of B_1 and B_2 out to the end of the overlap of the A_n's they lie in, i.e., for length

$$L = \min \left(h_1(n) - k_i(h_1(n-1)+S_1(n)) - j_i\right). \tag{3.4}$$

By Corollary 3.2 this amounts to showing that the independent concatenation more than 2^{8n} times of the distribution of names across A_{n-1} situated independently in n-boxes at positions $0 \leqq \sigma_1 \leqq S_1(n)$ is close in $\bar{d}$ to itself shifted by $|j_1 - j_2|$ places.

This pairing of coordinates along the distributions breaks into overlaps of n-boxes, half of which are longer than $\frac{1}{2}(h_1(n-1)+S_1(n))$. On these longer overlaps, as $S_1(n) \geqq 2^{16n}(h_1(n-2)+S_1(n))$, we can pair all but 2^{-16n} of the positions σ_1 in the first n-box with σ_1' in the second n-box so that in this overlap $(n-1)$-boxes across the name from A_{n-1} align with each other. Thus, on these sections we can match the two distributions within 2^{-16n} in $\bar{d}$. On the remaining sections, less than half the coordinates, the positions of $(n-1)$-boxes are now fixed, but everything inside the $(n-1)$-boxes is independent of what we have done. On each of these sections re-apply the argument, but on $(n-1)$-boxes, to match within $2^{-16(n-1)}$ a further quarter of the coordinates. Continue down to $(n-2)$-boxes and so on. If $n \geqq 2(l+2)$, after $n/2$ iterations, on all but $2^{-(n/2)}$ of the coordinates, we have matched in $\bar{d}$ to within $2^{-(n/2)}$. All but 2^{-8n+1} of the past atoms B_i satisfy Corollary 3.3 and end 2^{-8n} from the edges of A_n. This shows (T_1, P_l) is v.w.b. and completes the following theorem.

THEOREM 3.1. *The transformation T_1 is isomorphic to a Bernoulli shift.*

PROOF. By the above, (T_1, P_l) is v.w.b. for all l. The P's refine to the full algebra of measurable sets. By Ornsteins Monotone Theorem (2), T_1 is Bernoulli.

Our next task is to compute $h(T_1)$, the entropy of T_1. As we have no generator for T_1, it is again necessary to compute $h(T_1, P_l)$, and use the fact that $h(T_1) = \lim_{l\to\infty} h(T_1, P_l)$.

By our argument that (T_1, P_l) is v.w.b. we know that for large enough n, (T_1, P_l)

is as close as we like in $\bar{d}$ and, hence, entropy to the independent concatenation of the distribution of T_1, P_l-names across A_{n-1} in n-boxes situated independently at positions $0 \leqq \sigma_1 \leqq S_1(n)$. The entropy of this latter process is

$$\lim_{N\to\infty} \frac{1}{N}\left[\ln(h_1(n-1)+S_1(n))\left(\left(S_1(n)2^{h\,(\text{dist. across } A_{n-1})}\right)\left[\frac{N}{h_1(n-1)+S_1(n)}\right]\right)\right]$$

(3.5)

$$= \frac{h\,(\text{dist. across } A_{n-1})}{h_1(n-1)+S_1(n)}.$$

The distribution of names across A_{n-1} is the independent concatenation of names across A_{n-2}, situated in boxes at positions $0 \leqq \sigma_1 \leqq S_1(n-1)$. Hence,

$$\begin{aligned} h\,(\text{dist. across } A_{n-1}) &= N_1(n-1)\ln(S_1(n-1)\cdot 2^{h\,(\text{dist. across } A_{n-2})}) \\ &= N_1(n-1)\ln(S_1(n-1)) + N_1(n-1)h\,(\text{dist. across } A_{n-2}). \end{aligned}$$

So (3.5) is equal to

$$\frac{N_1(n-1)\ln(S_1(n-1))}{h_1(n-1)+S_1(n)} + \frac{N_1(n-1)h\,(\text{dist. across } A_{n-2})}{h_1(n-1)+S_1(n)}$$

(3.6)

$$= \frac{h_1(n-1)}{h_1(n-1)+S_1(n)}\left(\frac{\ln(S_1(n-1))}{h_1(n-2)+S_1(n-1)} + \frac{h\,(\text{dist. across } A_{n-2})}{h_1(n-2)+S_1(n-1)}\right).$$

Continuing inductively down to $l+2$, (3.6) is equal to

$$\left(\prod_{i=l+2}^{n-1}\frac{h_1(i)}{h_1(i)+S_1(i+1)}\right)\left(\frac{h\,(\text{dist. across } A_{l+1})}{h_1(l+1)+S_1(l+2)}\right)$$

(3.7)

$$+\sum_{i=l+2}^{n-1}\frac{h_1(i)}{h_1(i)+S_1(i+1)}\left(\frac{\ln(S_1(i))}{h_1(i+1)+S_1(i)}\right).$$

Thus,

$$h(T_1,P_l) = \left(\prod_{i=l+2}^{\infty}\frac{h_1(i)}{h_1(i)+S_1(i+1)}\right)\frac{h\,(\text{dist. across } A_{l+1})}{h_1(l+1)+S_1(l+2)}$$

(3.8)

$$+\sum_{i=l+2}^{\infty}\frac{h_1(i)}{h_1(i)+S_1(i+1)}\left(\frac{\ln(S_1(i))}{h_1(i-1)+S_1(i)+1}\right).$$

By (1.1),

$$\prod_{i=0}^{\infty}\frac{h_1(i)}{h_1(i)+S_1(i+1)} > 0$$

and

$$\sum_{i=0}^{\infty} \frac{h_1(i)}{h_1(i)+S_1(i+1)} \left(\frac{\ln (S_1(i))}{h_1(i-1)+S_1(i)} \right) < \infty$$

so

$$h(T) = \lim_{l\to\infty} h(T_1, P_l) = \lim_{l\to\infty} \frac{h \text{ (dist. of } P_l \text{ across } A_{l+1})}{h_1(l+1)+S_1(l+2)}. \tag{3.9}$$

Now to show that this value can be arbitrary.

Ignoring the small parts of A_{l+1} near the borders of the boxes, h(dist. of P_l across A_{l+1}) is close to $\ln [(h_2(l)+S_2(l+1))(S_1(l+1)S_2(l+1))^{N_1(l+1)}]$ and so (3.9) is equal to

$$\begin{aligned} &\lim_{l\to\infty} \left(\frac{\ln (h_2(l)+S_2(l+1))}{h_1(l+1)+S_1(l+2)} + \frac{N_1(l+1) \ln (S_1(l+1)S_2(l+1))}{h_1(l+1)+S_1(l+2)} \right) \\ &\qquad = \lim_{l\to\infty} \frac{\ln (S_2(l+1))}{h_1(l)}. \end{aligned} \tag{3.10}$$

By (3.2),

$$\frac{\ln (S_2(l+1))}{h_1(l)} = \frac{\ln [K_{l+1}(S_1(l)S_2(l)^{N_1(l)})(h_2(l-1)+S_2(l))]}{h_1(l)}$$

where all we require of K_{l+1} is that it is $\geqq 2^{16(l+1)}$ for large enough l. Notice this is independent of how large $N_2(l)$ is. But this is

$$\frac{\ln (K_{l+1}) + \ln (h_2(l-1)+S_2(l))}{h_1(l)} + \frac{\ln (S_1(l))}{h_1(l-1)+S_1(l)} + \frac{\ln (S_2(l))}{h_1(l-1)+S_1(l)}.$$

Now

$$\frac{\ln (h_2(l-1)+S_2(l))}{h_1(l)} + \frac{\ln (S_1(l))}{h_1(l-1)+S_1(l)} \leqq \frac{1}{2^{16l}}.$$

Thus

$$\frac{n(S_2(l+1))}{h_1(l)} = \left(\frac{h_1(l-1)}{h_1(l-1)+S_1(l)} \right) \left(\frac{n(S_2(l))}{h_1(l-1)} \right) + \frac{\ln (K_{l+1})}{h_1(l)} + \alpha_l$$

where $\alpha_l < 1/2^{16n}$ depends only on parameters up to the l-th, and $(\ln (K_{l+1}))/h_1(l)$ can be as small as we like up to a point, after which it can be any value $\geqq 2^{-16l}$. Thus, choosing the K_{l+1} properly, and $h_1(0)$, $h_2(0)$, $S_1(1)$ and $S_2(1)$, $\lim_{l\to\infty} [(\ln (S_2(l)))/h_1(l-1)]$ can be made to converge to any value we like in $(0,\infty]$, and this can be done even with $N_2(n)$ growing as rapidly as we like. This completes the following result.

THEOREM 3.2. *The entropy of T_1 can be chosen arbitrarily, even with the parameter $N_2(n)$ growing as rapidly as we like.*

IV. Forcing $C(T_1, T_2^j)$, $j \neq 0$ to be trivial

This argument is essentially a direct translation into Z^2 of the proof for a similar Z action in [1] and [4]. Thus, at points in the argument where standard techniques are used we will be brief.

At this point what remains at our disposal in the construction is that $N_2(n)$ can be made as large as we like, and the sequence $L_{j_1}, L_{j_2}, \cdots, L_{j_{N_2(n)}}$ of elements of $\mathcal{L}(n)$ (= the set of possible vertical cycles through the various rows of n-boxes $A^l_{n-1} = A^{\sigma_1^1, \sigma_2^2}_{n-1} \circ \cdots \circ A^{\sigma_1^{N_1}, \sigma_2^{N_1(n)}}_{n-1}$) remains to be chosen. We will choose these parameters to force $C(\{T_1, T_2^j\}) = \{T_1^l \circ T_2^k \mid l ; k \in Z\}$ for any $j \neq 0$.

The idea here is to choose $N_2(n)$ and $L_{j_1}, \cdots, L_{j_{N_2(n)}}$ inductively so that any two different rectangular sections of A_n of sufficient size have T_1, T_2, P-names bounded apart by some fixed constant in $\bar{d}$. This forces any finite coding of a T_1, T_2, P-name to within this bound in $\bar{d}$ of another T_1, T_2, P-name to be simply a shift of this name by some vector (l, k). This is enough for $C(\{T_1, T_2\})$ and a little more care gives the result for $C(\{T_1, T_2^j\})$.

LEMMA 4.1. *The parameters $N_2(n)$, $L_{j_1}, \cdots, L_{j_{N_2(n)}}$ can be chosen so that for any t, l, i and k, $N_2(n) \geqq t > N_2(n) 2^{-16n}$,*

$$0 < l < n!,$$

$$0 < i \leqq N_2(n) - tl \quad \textit{and}$$

$$0 < k < i,$$

the sequence of pairs

$$(L_{j_i}, L_{j_{i-k}}), \quad (L_{j_{i+l}}, L_{j_{i+l-k}}), \cdots, (L_{j_{i+tl}}, L_{j_{i+tl-k}})$$

is within 2^{-16n} of uniformly distributed over $\mathcal{L}(n) \times \mathcal{L}(n)$, and the sequence $L_{j_i}, L_{j_{i+l}}, \cdots, L_{j_{i+tl}}$ is within 2^{-16n} of uniformly distributed over $\mathcal{L}(n)$.

PROOF. The set $\mathcal{L}(n)$ is now fixed and contains s elements. Consider the set of all sequences of length N_2 of elements from $\mathcal{L}(n)$, with the counting measure. Fix values t, l, i and k as above. Look at the sequence of pairs

$$(L_{j_i}, L_{j_{i-k}}), \quad (L_{j_{i+l}}, L_{j_{i+l-k}}), \cdots, (L_{j_{i+tl}}, L_{j_{i+tl-k}}).$$

This can be broken into two disjoint subsequences, each at least $t/3$ long, in each of which no L_j occurs more than once. Now what is the probability on such a subsequence that the pairs are not within 2^{-16n} of uniformly distributed? Each term in the subsequence is independent of all the others, so this probability computes to be bounded by

$$2^{N_2 \varepsilon_{N_2}} 2^{(h(S^{-2}(1-2^{-16n}), S^{-2}(1+2^{-16n}(S^2-1)^{-1}), \cdots, S^{-2}(1+2^{-16n}(S^2-1)^{-1})) - h(S^{-2}, S^{-2}, \cdots, S^{-2}))N_2 2^{-16n} 3^{-1}} \tag{4.1}$$

where $\varepsilon_{N_2} \xrightarrow[N_2]{} 0$. But

$$h(S^{-2}(1-2^{-16n}), S^{-2}(1+2^{-16n}(S^2-1)^{-1}), \cdots, S^{-2}(1+2^{-16n}(S^2-1)^{-1})) - h(S^{-2}, S^{-2}, \cdots, S^{-2}) < 0 .$$

Thus, there are constants C and $\gamma < 0$ so that (4.1) is bounded by $C \cdot 2^{\gamma N_2}$.

Now the probability that there exists t, l, i and k so that the desired condition on pairs is not satisfied is thus at most

$$2 \cdot n! \cdot N_2^3 \cdot C\, 2^{\gamma N_2} . \tag{4.2}$$

This goes to zero as $N_2 \to \infty$. Hence, if N_2 is large enough, we can find the necessary sequence. The second condition follows from the first.

Choose $N_2(n)$, $L_{j_1}, \cdots, L_{j_{N_2(n)}}$ to satisfy this lemma.

Lemma 4.2. *There is a fixed constant $\alpha > 0$ so that any two different rectangular subarrays of at least 2^{-8n} of A_n are at least α apart in $\bar{d}$.*

Proof. In the overlap of two copies of A_0 which do not line up exactly there is always a $\bar{d}$ difference of at least $2(h_1(0) + h_2(0) - 1)/h_1(0)h_2(0)$.

Suppose any time we look at two rectangular sections of A_n which occupy at least 2^{-8n} of A_n, they differ in $\bar{d}$ by α_n. We want to bound α_{n+1} in terms of α_n. Consider the $\bar{d}$ distance between two rectangular sections of at least $2^{-8(n+1)}$ of A_{n+1}. All but at most $3 \cdot 2^{-8n}$ of this double name is made up of rectangular overlaps of at least 2^{-8n} of copies of A_n. A copy of A_n in one name overlaps at most four copies in the other. Consider separately each subset of one quarter of the overlaps which are between the A_n's in $(n+1)$-boxes in A_{n+1} at positions (i, j) and $(i + l, j + l')$, l and l' fixed. There are various cases. (i) If $l < h_1(n) + S_1(n+1)$ and $l' < h_2(n) + S_2(n+1)$, then these overlaps are between copies of A_n in exactly the same $(n+1)$-boxes in A_{n+1}. In these boxes the shifts (σ_1, σ_2) are identical. As $l, l' \neq 0$, in such a set of overlaps of A_n's, the $\bar{d}$ distance between the two rectangles is at least α_n. (ii) If $l \geqq h_1(n) + S_1(n+1)$ but $l' < h_2(n) + S_2(n+1)$, the overlaps are between different $(n+1)$-boxes in the same row across A_{n+1}. As $2^{-8(n+1)}$ of A_{n+1} must overlap parts of $2^{-8(n+1)}$ of the full cycles of possible rows, in this subset of overlaps we will see all possible positions (σ_1, σ_2) matched with (σ'_1, σ'_2), each with within $2^{-16(n+1)}$ of the same probability. At most $(S_1(n+1)S_2(n+1))^{-1}(1+2^{-16n+1})$ of these, then, can have their copies of A_n matching exactly. On the rest there is again a $\bar{d}$ distance of at least α_n. (iii) If $l' > h_2(n) + S_2(n+1)$ but $< (h_2(n) + S_2(n+1))(S_1(n+1)S_2(n+1))^{N_1(n+1)}$ then a part of the overlapping $(n+1)$-boxes still both lie in the same A^j_{n+1}. Look at only this part of the overlap. Now consider a subset of overlaps where each of the $(n+1)$-boxes in the first rectangle lies at the same relative position as all the others in the A^j_{n+1} containing it. A rectangle $2^{-8(n+1)}$ of A_{n+1} contains parts of $2^{-8(n+1)}$ of the A^j_{n+1}. On this sequence, by our choice of parameters in Lemma 4.1, each L_j is given about equal density. Thus, for each of the sets of $(n+1)$ boxes we consider, only at most $(S_1(n+1)S_2(n+1))^{-1}(1+2^{-16n})$ can have copies of A_n matching exactly. On the rest we get a $\bar{d}$ distance of at least α_n. (iv) On the remaining boxes in case (ii), or if $l' > (h_2(n) + S_2(n+1))(S_1(n+1)S_2(n+1))^{N_1(n+1)}$, the $(n+1)$-boxes which overlap each other lie in different cycles $A^{j_i}_{n+1}$, $A^{j_i+k}_{n+1}$. Consider a subset of overlaps where k is fixed. This splits the overlaps of $(n+1)$-boxes into two sets, each a sequence of pairs with types of the form $(L_{j_i}, L_{j_{i+k}})$, $(L_{j_{i+1}}, L_{j_{i+1+k}}) \cdots (L_{j_{i+t}}, L_{j_{i+t+k}})$, $t \geqq 2^{-8n}N_2(n)$. By our choice, from Lemma 4.1, these pairs are nearly equidistributed over all the possible pairs. This says that at most $(S_1(n+1)S_2(n+1))^{-1}(1+2^{-16n})$ of the overlaps have copies of A_n matching exactly. On the rest they are again α_n apart in $\bar{d}$.

Combining cases (i) through (iv) we see that

$$\begin{aligned}\alpha_{n+1} &\geqq \alpha_n\,(1 - 3\cdot 2^{-8n} - (S_1(n+1)S_2(n+1))^{-1}(1 - 2^{-4n} - 2^{-8n}))\\ &\geqq \alpha_n\,(1 - 3\cdot 2^{-8n} - 2^{-32n}\,(1 - 2^{-4n} - 2^{-8n})).\end{aligned}$$

This is enough to bound α_n uniformly away from 0, and completes the proof.

THEOREM 4.1. *If S commutes with both T_1 and T_2^j, $j \neq 0$, then $S = T_1^l \circ T_2^k$ for some l and k.*

PROOF. To start with, assume $j = 1$. Approximate S to within $\alpha^4 \cdot 10^{-12}$ in $\bar{d}$ by a finite coding of T_1, T_2, P-names, i.e., the set in P that $S^{-1}(w)$ lies in is determined all but $\alpha^4 \cdot 10^{-12}$ of the time by which set in $\bigvee_{p=-N}^{N} \bigvee_{q=-N}^{N} T_1^p \circ T_2^q(P)$ w belongs to, for some N large enough. Look at the occurrences in the T_1, T_2, P-name of a point ω of copies of A_n, where $h_1(n), h_2(n) > 4\cdot 10^{12} N \cdot \alpha^{-4}$. All but $\alpha^2 \cdot 10^{-6}$ of these finitely code to within $\alpha^2 \cdot 10^{-6}$ in $\bar{d}$ of their true image name under S. All the finite codings are within $\alpha^4 \cdot 10^{-12}$ of identical in $\bar{d}$. Copies of A_n

occupy all but 2^{-16n} of both the image and preimage names. Hence, an A_n preimage covers, most of the time, pieces of copies of A_n in the image. These pieces, four at most, are rectangular, and most of the time, larger than 2^{-8n} of an A_n. By Lemma 4.1, all such images, where the preimage codes well, are of exactly the same part of an A_n. Thus, on all but $(\alpha^2 \cdot 10^{-6} + 2^{-8n})$ of all but $\alpha^2 \cdot 10^{-6} + 2^{-8n}$ of the A_n, the rectangular pieces in the image of this A_n are the same. Now look at an A_{n+1} which has a fraction $\alpha^2 \cdot 10^{-6} + 2^{-8n}$ of the A_n's in it coding to the same rectangular pieces. If there is more than one rectangular piece in this overlap, our choice in Lemma 4.1 says that, if n is large enough, this large a fraction of pairs of n-boxes a fixed vector apart cannot have their copies of A_n a fixed vector apart. We conclude that an A_n which finitely codes this well codes to a shift of an A_n by some vector (l_n, k_n), $l_n < (\alpha^2 \cdot 10^{-6} + 2^{-8n}) h_1(n)$, $k_n < (\alpha^2 \cdot 10^{-6} + 2^{-8n}) h_2(n)$.

If l_n and k_n are constant, we are done. Look at a copy of A_{n+1} which codes to a shift of itself by (l_{n+1}, k_{n+1}) and for which all but $\alpha^2 \cdot 10^{-4}$ of the copies of A_n in it code to shifts of themselves by (l_n, k_n). As a shift by (l_n, k_n) of A_n covers all but $(\alpha^2 \cdot 10^{-6} + 2^{-8n})$ of an A_n, we conclude that a shift of A_{n+1} by $T_1^{l_n - l_{n+1}} \cdot T_2^{k_n - k_{n+1}}$ differs from itself in $\bar{d}$ by at most $(\alpha^2 \cdot 10^{-4} + \alpha^2 \cdot 10^{-6} + 2^{-8n} + 2\alpha^2 \cdot 10^{-6}) < \alpha$. Hence $l_{n+1} = l_n$ and $k_{n+1} = k_n$ by Lemma 4.2.

To extend this to larger values of j, consider an $S \in C(T_1, T_2^j)$ and finite codes of T_1, T_2^j, $\bigvee_{q=0}^{j-1} T_2^q(P)$ names which approximate S in $\bar{d}$. Such names cross a copy of A_n in j different ways, each one of which might code to a different part of an A_n. As before, though, a fixed way of crossing A_2 must, most of the time, code to precisely the same part of an A_n.

Now look at an A_{n+1} which codes well, most of its A_n's coding well. We restrict ourselves to subsequences $A_{n+1}^{j_i}, A_{n+1}^{j_i+j}, \cdots, A_{n+1}^{j_i+ij}$, of the cycles of rows in the preimage name of the overlap, all crossed in exactly the same way by T_2^j. If the image name was A_{n+1} shifted by more than an $(n+1)$-box, we are looking at overlaps of pairs of cycles of types (L_{j_i}, L_{j_i+l}), $(L_{j_i+j}, L_{j_i+j+l}), \cdots, (L_{j_i+ij}, L_{j_i+ij+l})$. If $n > j$, by our choice this sequence is nearly uniformly distributed over the space of all pairs. Hence, at most $(S_1(n+1)S_2(n+1))^{-1}(1 + 2^{-16n})$ of the A_n's in this overlap could code well. This is a conflict. Hence, the shift is by less than an $(n+1)$-box, and most A_n's, no matter how crossed by T_2^j, must code to the same shift of an A_n. As in the earlier case we can now show that the shift does not vary in n and so $S = T_1^l \circ T_2^k$ for some l and k.

This now finishes our construction and the proof of our main result.

THEOREM 4.2. *If T is a Bernoulli shift then* $C(C(T)) = \{T_i \mid i \in Z\}$.

Here are some further questions on centralizers that might be approached by similar constructive techniques and coding arguments.

(a) What is $C(T)$ for T a Bernoulli shift? We know $S \in C(T)$ implies S is weak mixing. Is this enough, i.e., for any weak mixing S is there an S' isomorphic to S in $C(T)$? Or more strongly, for any group G of weakly mixing transformations, is there a G' isomorphic to G in $C(T)$?

(b) One can argue, just as in Theorem 3.1, that for any j, $i \neq 0$, $T_1^i \circ T_2^j$ is a Bernoulli shift. Thus, the only elements of this Z^2 action that are not Bernoulli are the powers of T_2. These are, in fact, zero entropy transformations. Is it possible to build a Z^2 action, all of whose elements are Bernoulli, but whose centralizer is itself?

(c) Is it possible for other than finite rank transformations to have the property of commuting only with their powers? For example, can one get positive entropy?

(d) What is the centralizer of an Ornstein–Shields K-automorphism? Can such always be written as an automorphism of a complemented Bernoulli factor and a shift of the complement.?

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